“Trinomial Cube” Puzzles

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Objective

The component *pieces* of all ‘classic’ 3 x 3 x 3 (and 4 x 4 x 4 etc.) cube puzzles that I have seen are, basically, combinations of *identical* ‘unit’ cubes. For example, an ‘L’ shaped piece in a ‘classic’ 3 x 3 x 3 puzzle might combine 3 ‘unit’ cubes for the vertical part of the ‘L’ shape and a further single ‘unit’ cube for the ‘foot’ of the ‘L’. The components of the puzzle pieces —whatever the shape — are always identical ‘unit’ cubes! For a 3 x 3 x 3 puzzle (without ‘holes’) 27 ‘unit’ cubes must be employed to construct the full set of puzzle pieces.

This observation begs the question “*Why is it so?*” Is it possible to devise a new class of cube puzzle where the components of each puzzle *piece* are *rectangular solids* — including but not limited to simple cubes?

This article describes the development of such a ‘generalised’ version of the ‘classic’ 3 x 3 x 3 (and 4 x 4 x 4 or 5 x 5 x 5) ‘unit’ cube puzzles — where the component shapes / elements are rectangular solids (including cubes).

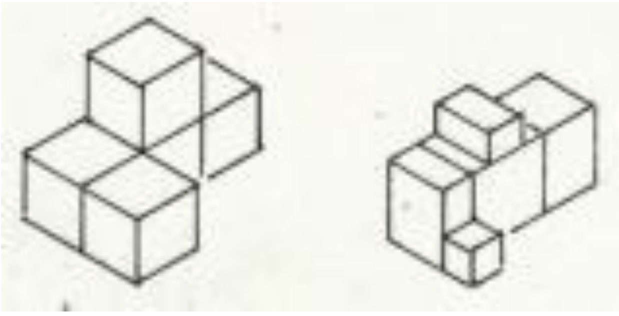
Background

All 3D puzzles — excepting those using pyramids, spheres etc. — are based on ‘unit’ cubes.
Let the side of a ‘unit’ cube be ‘a’ (e.g. 25mm).
Then the area of each face of the ‘unit’ cube is a x a (= a^2) and the volume a x a x a (= a^3).
Twenty-seven such ‘unit’ cubes may be combined to form a (larger) cube of side = 3a.
The volume of this (larger) cube = $(3a)^3 = 27a^3$.
Conventional 3x3x3 puzzles (or 4 x 4 x4 puzzles) combine (or ‘join’) ‘unit’ cubes in various ways.

A ‘classic’ 3 x 3 x3 puzzle ‘Piece’ is shown at left in the following diagram.
This typical ‘Piece’ consists of 5 (identical) ‘unit’ cubes or ‘components’) each of side ‘b’.

It should be possible to develop 3 x 3 x 3 cube puzzles where the components —instead of all being ‘unit’ cubes — have dimensions drawn from the set ‘a’, ‘b’, ‘c’ —where a < b < c and where ‘b’ and ‘c’ are not integral multiples of ‘a’. Such a puzzle ‘Piece’ is shown on the right in the diagram.

The two puzzle ‘Pieces’ shown in the diagram are isomorphic.
If ‘a’ is expanded (in size) to equal ‘b’ and ‘c’ shrunk (in size) to also equal ‘b’ then the second Element in the diagram ‘morphs’ to equal (or become) the first Element.



5 x ‘Elements’ $b^3 + b^3 + b^3 + b^3 + b^3$

5 x ‘Elements’ $a^3 + abc + a^2b + bc^2 + b^2c$

Theory

The expansion of $(a + b + c)^3$ is

$$= a^3 + 3a^2b + 3a^2c + 3ab^2 + 3ac^2 + 6abc + 3b^2c + 3bc^2 + b^3 + c^3 \qquad (1)$$

This expansion uses the (mathematical) ‘Trinomial Theorem’!
The 10 terms in the expansion contain, in total, 27 items (or ‘pieces’).
Each ‘piece’ in the expansion may be visualised as a simple 3D solid!
For example, the piece ‘a^2b’would have physical dimensions …’a x a x b’.
The term $(a + b + c)^3$ represents the ‘volume’ of a cube of side $(a + b + c)$.

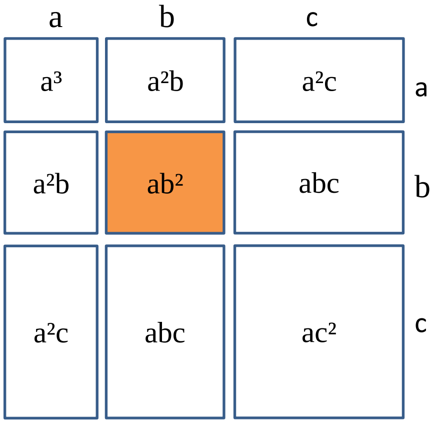
The 27 pieces specified in Equation (1) may also be ‘joined’ in pairs, triplets etc.
‘Joined’ pieces may also be used to form the 3 x3x3 puzzle.
All 27 ‘pieces’ must be used —if the completed puzzle is to be devoid of ‘holes’.

This theory underpins the new puzzles class —the “Trinomial Cubes”.

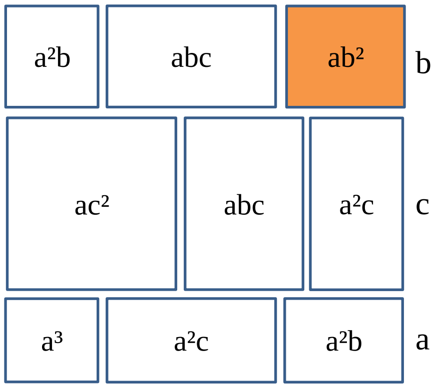
Standard Layout

The 27 ‘pieces’ of Equation (1) may be arranged in 3 ‘Layers’
Each Layer consists of nine (9) fundamental ‘pieces’.
Each Layer forms a square with side = $(a + b + c)$.
Three Layers stack to form a cube of side $(a + b +c)$ and volume $(a + b + c)^3$.
The 3 Layers may be named ‘a’, ‘b’ and ‘c’.
Layers ‘a’, ‘b’ and ‘c’ have vertical ‘dimensions’ of ‘a’, ‘b’ and ‘c’ respectively.

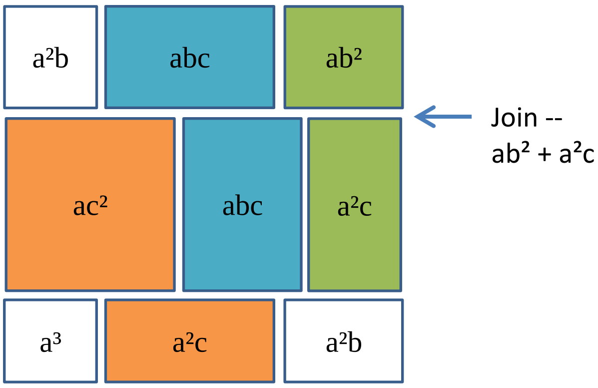
Level 'a' – Standard



Level 'a' – Rearranged



It is also possible to 'join' two (or more) 'pieces' in a Layer.
In the diagram colours show 3 'joins' in a (rearranged) Layer 'a'.
Thus piece 'ab^2' is 'joined' (glued) to piece 'a^{2}c'.
The other two 'joins' are 'ac^2' to 'a^{2}c' and 'abc' to 'abc'.
In this arrangement 3 'pieces' are left unjoined.
Many such arrangements are possible!



Design Conventions

My personal design conventions are:

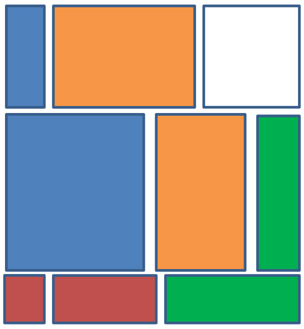
1. $a=20mm < b=35mm < c=45mm$ — $(a + b + c) = 100mm$
   Note that, for example, 'b' is not an integral multiple of 'a'.
2. Layer order — Layer 'c' (lowest), Layer 'a' (middle), Layer 'b' (highest).
   The smallest / thinnest Layer is 'sandwiched' between two thicker Layers.

Three (3) possible example Trinomial Cube Puzzles are outlined below.
All have been constructed using Tasmanian ('Tassie') Oak.

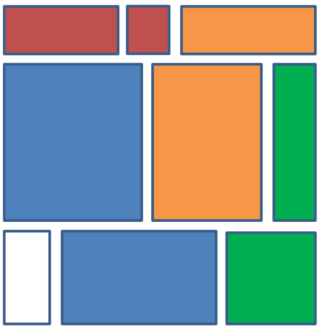
Trinomial Cube #1

Features --

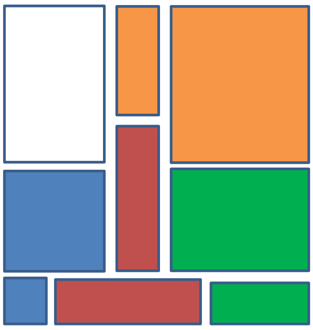
1. Each Layer features a different set of ‘joins’.
2. There are no ‘joins’ between / across Layers.
   (Each Layer is independent of the other Layers.)
3. Colours indicate ’joined’ ‘pieces’.
4. No colour indicates an ‘unjoined’ ‘piece’.



Level ‘c’ (Bottom)



Level ‘a’ (Middle)



Level ‘b’ (Top)

All example puzzles have been made from wood. There are no colours!
It can take a puzzler time to realise that there are Layers, sort the Layers and then assemble.
In the completed puzzle the Layers are quite visible.

More complicated joins are possible and indeed desirable.

1. Joins between ‘pieces’ in adjacent Layers. (Trinomial Cube #2)
2. Joins between ‘pieces’ in all three Layers. (Trinomial Cube #3)

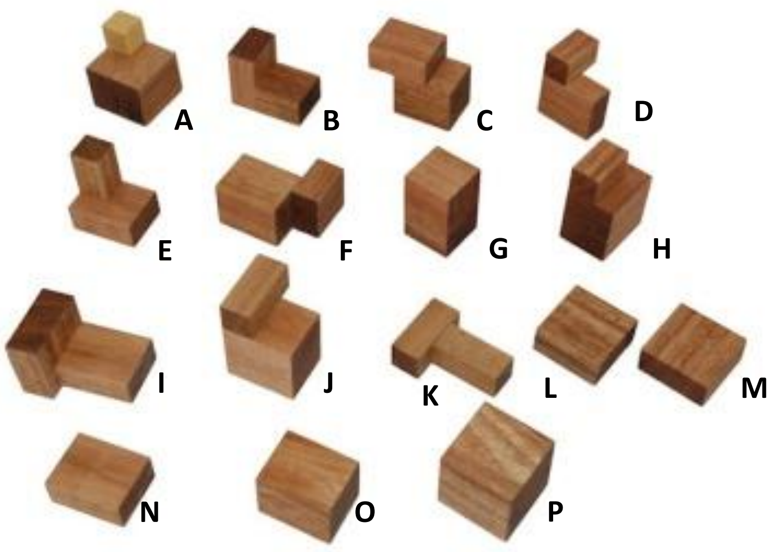
These (and other) “Trinomial Cube” puzzles can be quite challenginng!

Trinomial Cube #2

Features --

1. Layers 'a', 'b' and 'c' all possess (different) joined 'pieces'.
   Some 'pieces' in each Layer are not joined.
2. Joins between Layer 'c' (lowest) and Layer 'a' (middle).
   Joins between Layer 'a' (middle) and Layer 'b' (highest).
3. No joins across all three Layers.
   This is a feature of Trinomial Cube #3!

The resulting 16 pieces may be assembled into a cube of side (a + b + c) = 100mm.
The Layers (Layer 'b' bottom, Layer 'a' middle and Layer 'b' top) in the assembled puzzle are distinguished.



A ... $a^3 + bc^2$	**B** ... $abc + ab^2$	**C** ... $abc + b^2c$
D ... $abc + a^2b$	**E** ... $abc + a^2b$	**F** ... $b^2c + ab^2$
G ... $ab^2 + b^3$	**H** ... $a^2b + bc^2$	**I** ... $ac^2 + abc$
J ... $a^2c + b^2c$	**K** ... $a^2c + a^2c$	**L** ... ac^2
M ... ac^2	**N** ... abc	**O** bc^2
P ... c^3	**Total** – 16 Elements (27 'pieces')	

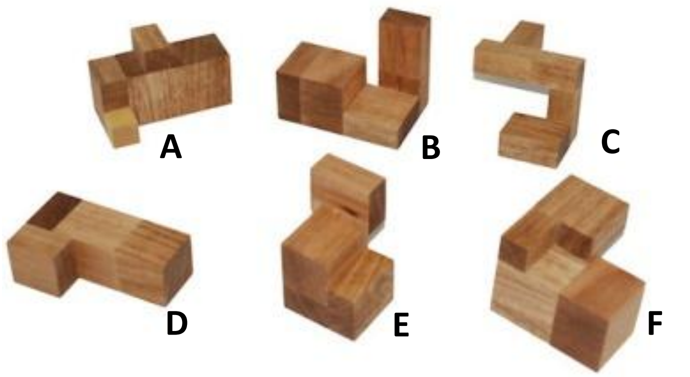
Solution :
Uses "Puzzle Will Be Played" website conventions!

Bottom (Layer 'c')	*Middle* (Layer 'a')	*Top* (Layer 'b')
N K M	**A K D**	**A F D**
I O J	**I E J**	**C E F**
P H L	**B H G**	**B C G**

Trinomial Cube #3

Features --

1. 'Joins' across all three Layers.
2. Six Elements (27 pieces).



A = $a^3 + abc + bc^2 + a^2b + b^2c$ --- 5 'pieces'

B = $2ac^2 + ab^2 + abc + bc^2$ --- 5

C = $a^2b + 3a^2c + abc$ --- 5

D = $2ab^2 + b^2c + b^3$ --- 4

E = $3abc + b^2c$ --- 4

F = $a^2b + ac^2 + bc^2 + c^3$ --- 4 Total 27

Solution :

Uses "Puzzle Will Be Played" website conventions!

Bottom (Layer 'c')	*Middle* (Layer 'a')	*Top* (Layer 'b')
E C B	**A C C**	**A A A**
E E B	**F E C**	**D A C**
F F B	**F B B**	**D D D**

Extensions

In principle, any existing 3x3x3 cube puzzle may be recast using the 'elements' (or pieces) identified in Equation (1) above. Exceptions (if any) might include certain 'interlocking' puzzles and puzzles where rotations are required. In this sense the "Trinomial Cube" puzzles may be considered generalised versions of 'standard / conventional' 3x3x3 cube puzzles.

Expansion of the algebraic expression $(a + b + c + d)^3$ covers the common 4x4x4 cubes. There are just more 'pieces' and ever more sophisticated 'join' opportunities.

The theory is also readily extended to cover rectangular solids. That case is covered by the algebraic expression $(\Sigma a_i)\,(\Sigma b_j)\,(\Sigma c_k)$ where $\Sigma a_i \neq \Sigma b_j \neq \Sigma c_k$.

Unfortunately, powers greater than 3 — or, for that matter, fractional powers — do not lend themselves to visualisation and construction!

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